EUCLID-BOOK XII

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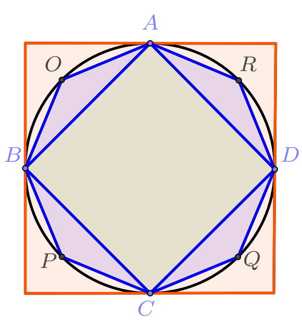
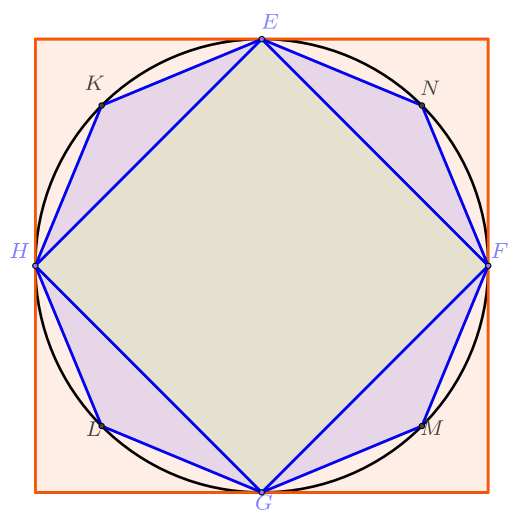
1. Proposition XII, 2

In book XII of the *Elements*, Euclid proves the following theorems:

- **El. XII, 1**. Similar polygons in circles are to one another as the squares from the diameters.
- **El. XII, 2**. Circles are to each other as the squares of their respective diameters.

XII, 2 can be immediately derived from XII, 1 if we imagine that a circle is an *infinitangular polygon*. However, Euclid does not proceed in this way. Euclid's proof of XII, 2 is indirect: it starts from the assumption that the ratio $ABCD : EFGH \neq Q(BD) : Q(HF)$, and derives a contradiction.





Euclid's indirect proof is also known as a *proof by exhaustion*, although the term "exhaustion" refers to the technique employed in the proof, which consists of "filling up" the circles with polygons. Euclid XII, 2 is a good example of a theorem rigorously proven according to a "Euclidean' ideal of rigour consisting in avoiding infinite or infinitesimal considerations.

However, the price to pay for this rigorous proof, as the early moderns knew well, was a certain prolixity and obscurity. As Tacquet noted in his *Elementa Geometriae*:

> Demonstrationes adeo sunt prolixae, ut Tyrones in desperationem plerumque conijciant

As said above, to prove proposition XII, 2, Euclid assumes that $Q(BD) : Q(HF) \neq ABCD : EFGH$. From this, he infers that there exists a magnitude S, such that: $Q(BD) : Q(HF) = ABCD : S$. Since $S \neq EFGH$, Euclid assumes that either $S < EFGH$, or $S > EGFH$. In the rest of the proof, Euclid shows that a contradiction follows from either case. Like other proofs in reductio style, the proof of XII, 2 rests on two assumptions:

- given three magnitudes, A, B, C, in which the ratio $A : B$ exists (i.e. A and B are homogeneous, according to Euclid, V, df. 4), there exists a fourth magnitude X, such that $A : B :: C : X$ (this is the assumption of the fourth proportional, which can be considered a kind of continuity axiom).
- Given two homogeneous magnitudes A and B, either $A < B$, or $A > B$, or $A = B$ (this assumption ensures that homogeneous geometrical magnitudes are always comparable, hence they form a complete order).

With Euclid, let us first consider the case where: $S < EFGH$. Let us inscribe a square $q(HF)$, with diagonals HF, EG, in the circle $EFGH$. We will have that: $q(HF) = \frac{1}{2}Q(HF)$. Since the circle $EFGH$ is included in the square $Q(HF)$, we will have that: $EFGH < Q(HF)$, and: $\frac{1}{2}EFGH < \frac{1}{2}Q(HF) = q(HF)$, This proves that the inscribed square $q(HF)$ is greater than half of the circle $EFGH$. The next step consists in constructing the series of triangles ENF, EMG ..., each obtained by halving arcs EF, FG, ..., as shown in figure.

For each triangle (e.g. ENF) it is possible to prove that it is greater than half of the corresponding sector of the circle in which it is inscribed (e.g. the sector ENF). The proof can be give for just one case as follows:

$$triangle(ENF) = \frac{1}{2}rect(EF)$$

$$sector(ENF) < rect(EF) \rightarrow \frac{1}{2}sector(ENF) < \frac{1}{2}rect(EF) = triangle(ENF)$$

$$\frac{1}{2}sector(ENF) < triangle(ENF)$$

By continuing bisecting the arcs, we can construct more and more triangles, for which we ca show that they are greater than half of the corresponding sectors using the same proof.

At this point, Euclid can rely on X, 1 as a lemma:

Theorem 1 (Elements, X, 1)**.** *if we subtract from a given magnitude a part that is more than half of the magnitude, and we do that at every step, after n-steps we will remain with a magnitude as small as we please.*

If the given magnitude is the circle $EFGH$, from which we subtract a series of polygons as described above (e.g. the square $q(HF)$ at step 1, and at each $n+1$ step, 2^n+1 triangles constructed as explained above), then we fall into the case of X, 1. Each polygon obtained at a given step n is always greater than half of the (remaining) circle. From this, we conclude that a quantity a can be obtained, after a certain number of steps, that is smaller than the given quantity $EFGH - S$, i.e. we shall have: $S < EFGH - a$. By construction, the magnitude $EFGH - a$ is a polygons, because it is the figure obtained by taken off 2^n+1 circular sectors from a given circle. Let us call this polygon P_2.

Let us assume that P_2 is the polygon $ENFMGLHK$. A similar polygon $ARDQCPBO$ (we can call it P_1 for brevity) can be constructed in the circle $ABCD$. We have, from XII, 1:

$$Q(BD) : Q(HF) = P_1 : P_2$$

From V, 11 we also have:

$$ABCD : S = P_1 : P_2$$

Using a property of proportion (permutation, Euclid V, 16) we conclude that: $ABCD : P_1 = S : P_2$

Since $P_1 < ABCD$, because it is inscribed in that circle, it results that P_2 is smaller that S. However, this is a contradiction, because we have claimed above that $S < EFGH - a$, i.e. $S < P_2$. Hence:

$$Q(BD) : Q(HF) \neq ABCD : S$$

with $S < EFGH$. In the same way, we prove that: $Q(HF) : Q(BD) \neq EFGH : S$, with $S < ABCD$.

In the second part of the proof, Euclid assumes that there exists a magnitude S, such that $S > EFGH$ and that: $Q(BD) : Q(HF) = ABCD : S$. From this proportion, it follows: $Q(HF) : Q(BD) : S : ABCD$. Since $S > EFGH$, we have, assuming the 4th proporitional, that $S : ABCD = EFGH : T$, where T is a magnitude homogeneous to $ABCD$, and $T < ABCD$. As a consequence, $Q(HF) : Q(BD) = EFGH : T$. However, we have proven above that $Q(HF) : Q(BD) \neq EFGH : S$, for a generic magnitude S with $S < ABCD$, hence a contradiction results.

Since the proportion $Q(BD) : Q(HF) = ABCD : S$ does not hold if $S < EFGH$ and if $S > EFGH$, it must hold in the case $EFGH = S$ (by the assumption of complete order). Hence $Q(BD) : Q(HF) = ABCD : EFGH$.

2. Some early modern elaborations

The process of "exhaustion" a figure by a sequence of inscribed polygons, implicit in Euclid's proof, became a fundamental technique that lay the groundwork of modern calculus. Several commentators of the *Elements* focussed on Euclid XII, 2 as a key result of the rigorous Euclidean way of handling infinitary processes.

2.1. Tacquet.

A. Tacquet was a jesuit professor, and wrote in 1654 a version of the Elements adapted for students. This proposition was singled out as particularly complex and hard to understand. For this reason, Tacquet sought for an easier way to prove it.

Tacquet adds to Euclid's book XII the definition of a sequence of inscribed or circumscribed figures tending to a given figure:

Definition. Magnitudes, whether inscribed or circumscribed to a certain figure, be they smaller or larger than the figure, are said to tend to the figure (*in figuram desinere*) if they can eventually differ from it by a quantity lesser than any given one, no matter how small.

On this basis, Tacquet then proves the following *porisma universale*:

if the figures inscribed in two magnitudes A, and B, tend to them, have the same proportion one to the other, then the figures A and B are also in the same proportion.

PROOF

We suppose that the figures inscribed in A and B are to the other in the same ratio $X : Z$. Let us suppose that it is not the case for A and B, hence $A : B \neq X : Z$. Let us suppose that $A : B > X : Z$. Hence there exists a quantity R, such that $R < A$, and: $R : B = X : Z$. Since we assumed that the figures inscribed in A and B tend to A and B, respectively, we shall have that there exist at least two figures, for instance, C and F, whose difference from A and B, respectively, is smaller than any given quantity. Let us assume that it is smaller than the given quantity $B - R$. Hence $A - C < B - R$. This inequality implies that $C > R$, so that: $C : B > R : B$. However we also assumed that:

$$R : B = X : Z$$

and, by hypothesis:

$$X : Z = C : F$$

since C and F are inscribed in A and B. It follows that the ratio $C : B$ is larger than the ratio $C : F$, and that $B > F$. But F is assumed to be inscribed in B, hence it cannot be larger. From this contradiction, it follows that:

$$A : B = X : Z.$$

On the basis of this *porisma universale*, to prove XII, 2 it is enough to prove that the polygons constructed by bisecting the arcs, as explained above, tend to the circles.

2.2. **Newton.** Tacquet's general porisma can be compared to Newton's lemma IV, Book I, *Principia*:

Lemma 2 (Newton)**.** If in two figures there should be inscribed (as above) two series of parallelograms, and if the number of both should be the same, and when the widths are diminished indefinitely, the last ratios of parallelograms in one figure should be individually the same as the parallelograms in the other figure; then the two figures are in the same ratio to one another.

3. James Gregory

In his *Vera Circuli et Hyperbolae Quadratura*, published in 1667, Gregory developed a theory of convergent series in order to solve the quadrature of the circle, the ellipse and the hyperbola (i.e. compute their areas, or the area of a sector as a function of the inscribed and circumscribed polygons). Gregory does not use Euclid XII, 2 but relies on X, 1 in an ingenuous way to prove that a sequence of inscribed and circumscribed polygons tends to a sector. Gregory's treatment remains Euclidean (or Archimedean, as no indivisible or infinitesimals are involved). We can say that Gregory pushes the Euclidean method to its limits.

One remarkable aspect of Gregory's treatise is the abstract definition of convergence, which generalizes Tacquet's concept of *desinare* as follows:

Definition 1 (series convergens)**.** (*VCHQ*, df. 9, p. 10) Given two successions of quantities $\{a_n\}$ and $\{b_n\}$, Gregory called "*convergent series*" a double sequence $\{a_n, b_b\}$, if the following conditions obtain (S and S' are two finite compositions):

There is a composition S such that,

$$\forall n, a_{n+1} = S(a_n, b_n)$$

There is a composition S' such that,

$$\forall n, b_{n+1} = S'(a_n, b_n)$$

and:

$$\forall n \mid b_{n+1} - a_{n+1} \mid < \mid b_n - a_n \mid$$

We notice that *convergece* is a property of a double sequence of quantities, in which each term is composed from the previous ones through a well define set of operations, which Gregory calls *compositio.* These operations are, for instance, the usual ones of arithmetic.

We also notice that the above conditions do not imply that the series tends to 0, that is, they don't imply convergence in the modern sense.

Afterwards, Gregory establishes that the series of polygonal figures inscribed and circumscribed to a sector of a circle (resp. ellipse, hyperbola) not only form a convergent series according to the definition above, but he also proves that the series of the differences tends to 0.

To prove this results, Gregory proves that the differences between the successive terms of the series $\{I_n\}$ and $\{C_n\}$, as n increases, becomes smaller, according to the following relation: $\forall n, (C_{n+1} - I_{n+1}) < \frac{1}{2}(C_n - I_n)$, in the case of the circle or the ellipse, and $\forall n(I_{n+1} - C_{n+1}) < \frac{1}{2}(I_n - C_n)$ in the case of the hyperbola.

Considering, for the sake of simplicity, only the case of the series approaching a sector of the circle or of an ellipse (in the case of the hyperbola, the following inequalities must be inverted), Gregory proceeds by giving a direct proof of the following inequality for the first pair of in- and circumscribed polygons:

$$C_1 - I_1 < \frac{1}{2}(C_0 - I_0) \tag{1}$$

He then generalizes this inequality to successive terms, on the strength of the recursive construction of the in- and circumscribed polygons. For the details of Gregory's proof, see my book, p. 59ff. Here I will only stress that the reason why Gregory relies on the inequality $C_1 - I_1 < \frac{1}{2}(C_0 - I_0)$ can be understood if we refer to Euclid, X, 1 mentioned above.

From Euclid's proposition X, 1we infer that:

Theorem 3. *If subtracting from a given magnitude M, a succession of quantities, such that each remainder is smaller than half of the previous one, we can obtain a succession of remainders that can be taken as small as we please.*

Thus at step 1, we will have a remainder R_1 such that $R_1 < \frac{1}{2}M$, at step 2, a remained $R_2 < \frac{1}{2}(M - M_1)$, ... at step n, we will have that $R_n < \frac{1}{2}(M - M_1 - M_2 - \cdots - M_{n-1}$. This situation applies to the case discussed by Gregory, as the differences $C_n - I_n$ between the n-th circumscribed and the inscribed polygons are less than half that the previous difference, hence the sequence of these differences can be continued until we find a difference as small as we please.

As a final result, Gregory proves that there exists a *last* term of the series, namely a pair of polygons, let us call them I_θ, C_θ, such that $C_\theta = I_\theta = \theta$ (θ is the area of the circular sector under consideration). Gregory proceeds with a reductio argument, and assumes: $\theta - I_\theta = Z$, where Z is an arbitrary quantity that differs from 0. Since the series of polygons is convergent, there exists a pair (I_n, C_n) such that $C_n - I_n < Z$. Hence $C_n - I_n < \theta - I_\theta$. This inequality implies: $I_n - I_\theta > C_n - \theta$. But this is a contradiction because the term $C_n - \theta$ is positive (C_n is circumscribed to the sector θ, hence larger than it), while the term $I_n - I_\theta$ is negative, since I_θ is the last inscribed polygon, hence $I_n < I_\theta$, Therefore we have: $C_\theta = I_\theta = \theta$.